EXAM MECHATRONICS (SOLUTIONS)
TUESDAY 24 JANUARY 2017, 9.00 - 12.00 h.

Name: ______________________________ StudentID: __________

- This exam consists of 23 pages with 6 open questions. Check if you have all pages.
- The answers to each question (including motivation) have to be placed in the answer boxes.
- Write neatly.
- Write your name and student number in all pages. The exercises will be collected separately.
- If you like, you can use and add additional paper, which needs to include your name and student number. Please provide separate papers for separate exercises. Hand in the exercises on separate piles.
- You can earn a maximum of 100 points at the exam. The amount of points spread over the exercises is 110 points, i.e., there are 10 bonus points to be earned.
- You will only get a grade if you have finalized the practical.
- This is a CLOSED book exam.
- Good luck!

Preliminaries

Across and Through variable table

Table 1.2. Ideal system elements (linear)

System type	Mechanical translational	Mechancial rotational	Electrical	Fluid	Thermal
A-type variable	Velocity, v	Velocity, Ω	Voltage, e	Pressure, P	Temperature, T
A-type element	Mass, m	Mass moment of inertia, J	Capacitor, C	Fluid Capacitor, C_f	Thermal capacitor, C_h
Elemental equations	$F = m\dfrac{dv}{dt}$	$T = J\dfrac{d\Omega}{dt}$	$i = C\dfrac{de}{dt}$	$Q_f = C_f\dfrac{dP}{dt}$	$Q_h = C_h\dfrac{dT}{dt}$
Energy stored	Kinetic	Kinetic	Electric field	Potential	Thermal
Energy equations	$\mathscr{E}_k = \frac{1}{2}mv^2$	$\mathscr{E}_k = \frac{1}{2}J\Omega^2$	$\mathscr{E}_e = \frac{1}{2}Ce^2$	$\mathscr{E}_p = \frac{1}{2}C_f P^2$	$\mathscr{E}_t = \frac{1}{2}C_h T^2$
T-type variable	Force, F	Torque, T	Current, i	Fluid flow rate, Q_f	Heat flow rate, Q_h
T-type element	Compliance, $1/k$	Compliance, $1/K$	Inductor, L	Inertor, I	None
Elemental equations	$v = \dfrac{1}{k}\dfrac{dF}{dt}$	$\Omega = \dfrac{1}{K}\dfrac{dT}{dt}$	$e = L\dfrac{di}{dt}$	$P = I\dfrac{dQ_f}{dt}$	
Energy stored	Potential	Potential	Magnetic field	Kinetic	
Energy equations	$\mathscr{E}_P = \dfrac{1}{2k}F^2$	$\mathscr{E}_P = \dfrac{1}{2K}T^2$	$\mathscr{E}_m = \frac{1}{2}Li^2$	$\mathscr{E}_k = \frac{1}{2}IQ_f^2$	
D-type element	Damper, b	Rotational damper, B	Resistor, R	Fluid resistor, R_f	Thermal resistor, R_h
Elemental equations	$F = bv$	$T = B\Omega$	$i = \dfrac{1}{R}e$	$Q_f = \dfrac{1}{R_f}P$	$Q_h = \dfrac{1}{R_h}T$
Rate of energy dissipated	$\dfrac{dE_D}{dt} = Fv = \dfrac{1}{b}F^2 = bv^2$	$\dfrac{dE_D}{dt} = T\Omega = \dfrac{1}{B}T^2 = B\Omega^2$	$\dfrac{dE_D}{dt} = ie = Ri^2 = \dfrac{1}{R}e^2$	$\dfrac{dE_D}{dt} = Q_f P = R_f Q_f^2 = \dfrac{1}{R_f}P^2$	$\dfrac{dE_D}{dt} = Q_h$

Note: A-type variable represents a spatial difference across the element.

The other analogy for linear systems as was treated in Control Engineering, and is useful for Euler-Lagrange modeling.

	Kinetic coenergy	**Potential energy**	**Rayleigh dissipation function**
Translation	$T^*(\dot{x}) = \frac{1}{2}m\frac{dx}{dt}^2$	$V(x) = \frac{1}{2}kx^2$	$\mathcal{D}(\dot{x}) = \frac{1}{2}b\frac{dx}{dt}^2$
Rotation	$T^*(\dot{\theta}) = \frac{1}{2}J\frac{d\theta}{dt}^2$	$V(\theta) = \frac{1}{2}k\theta^2$	$\mathcal{D}(\dot{\theta}) = \frac{1}{2}b\frac{d\theta}{dt}^2$
Electric	$T^*(\dot{q}) = \frac{1}{2}L\frac{dq}{dt}^2$	$V(q) = \frac{1}{2C}q^2$	$\mathcal{D}(\dot{q}) = \frac{1}{2}R\frac{dq}{dt}^2$

Canonical forms

The state-space representation for a given transfer function is not unique, i.e., there are infinite-number of possibilities to express a given transfer function in state-space form. However, there are several forms that can be helpful in the design of controller or observer. Let us consider the following general transfer function for single-input single-output system:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}. \tag{1}$$

For this transfer function, the state-space representation in **canonical observable form** is given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1}b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{aligned} \tag{2}$$

On the other hand, the state-space representation in the **canonical controllable form** is given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1}b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{aligned} \tag{3}$$

Z-transform.

Denote by $Z\{u(n)\}$ the Z-transform of discrete-time signal $u(n)$ where $n = 0, 1, \ldots$.

- Unit step signal $u(n)$: $Z\{u(n)\} = \dfrac{1}{1 - z^{-1}}$
- Time-shifting property: $Z\{u(n-k)\} = z^{-k}U(z)$

Transformation from s-domain to z-domain

- The bilinear transformation: $s \mapsto \dfrac{2}{T}\dfrac{1 - z^{-1}}{1 + z^{-1}}$
- The backward-Euler transformation: $s \mapsto \dfrac{1}{T}(1 - z^{-1})$

Optimal state feedback control design(LQR)

The Riccati equation, that is related to the optimal state feedback, reads as

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{4}$$

where Q and R are related to the cost function

$$J = \int_0^\infty x^T(\tau)Qx(\tau) + u^T Ru(\tau)d\tau. \tag{5}$$

The optimal state feedback controller is given by $u(t) = -R^{-1}B^T Px(t)$.

State observer design

For a state-space system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{6}$$

where x is the actual state and y is the measured signal, a state observer for such system has the following structure

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x}, \end{aligned} \tag{7}$$

where $\hat{x}$ is the estimated state and y is the corresponding estimated output.

Transfer function of time delay

For a time delay operator

$$y(t) = u(t - T) \tag{8}$$

where T is the delay time, its Laplace transform is given by

$$\frac{Y(s)}{U(s)} = e^{-sT}. \tag{9}$$

From this transfer function, the corresponding Bode plot is given by unity amplitude for all frequencies and the phase plot is linear with respect to the frequency, i.e.,

$$\phi(\omega) = -\omega T, \tag{10}$$

for all frequencies ω.

The first order Padé approximation of the delay transfer function $e^{-\omega T}$ is given by

$$e^{-\omega T} \approx \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}. \tag{11}$$

1. (10 TOTAL points) Consider the autonomous-driving car shown in fig. 1

Figure 1: Autonomous driving car being built by Google.

(a) (5 points) Identify three possible user demands, three possible functional requirements, and three possible design parameters.

Solution:

- User Demands:
  - Silent, fast, safe to collisions, economical, high efficiency...
- Functional requirements:
  - the car must recognize pass-by walkers; car must be able to regulate its velocity; energy efficiency should be of x hours or y kilometers; ...
- Design Parameters:
  - Car should have sensors to detect large objects to avoid collisions; the material of the car should be ecologically friendly; car should fit 2 persons comfortably,....

(b) (5 points) Referring to the mechatronic block diagram shown in fig. 2 identify at least 2 elements of the autonomous driving car for the each of the following items.

- measured variables
- manipulated variables
- reference variables
- sensors
- actuators
- man/machine interface
- energy supply

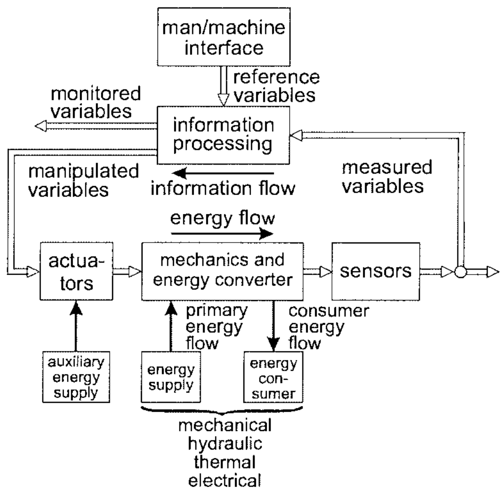


Figure 2: Mechatronics Block

Solution:

- measured variables: local position, global position, speed, humidity, temperature, light, ...
- manipulated variables: speed, local position, global position, light,...
- reference variables: speed, safety distance to avoid collision, ...
- sensors: GPS, sensors to detect obstacles, humidity sensor, temperature sensor,...
- actuators: motors/actuators to turn, move forwards/backwards/stop, activate wipers, etc.
- Man/machine interface: Computers on board, telemetry,...
- Energy supply: Batteries, Hydrogen, Solar, ...

2. (30 TOTAL points) Suppose that a transfer function of an industrial process from the input u to the measurement output y is given by

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s - 1}. \tag{12}$$

(a) (1 point) Is the system stable or unstable? Why?

Solution: The system is unstable because one root of the characteristic polynomial has positive real part.

(b) (1 point) Obtain the state-space representation of the system in the controllable canonical form.

Solution:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \tag{13}$$

(c) (12 points) Suppose that the production cost of the industrial process is given by

$$J = \int_0^{\infty} \left(3x_1^2(\tau) + 2x_1(\tau)x_2(\tau) + 2x_2^2(\tau) + 10u^2(\tau)\right) d\tau. \tag{14}$$

Show that the solution to the Riccati equation is given by

$$P = \begin{bmatrix} 35.69 & 21.40 \\ 21.40 & 13.41 \end{bmatrix} \tag{15}$$

and then obtain the optimal feedback controller that stabilizes the system and minimizes the cost J.

Solution: The Riccati equation reads as

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} \tag{16}$$

from which we obtain three equations

$$-\frac{1}{10}p_2^2 + 2p_2 = -3 \tag{17}$$

$$-\frac{1}{10}p_2 p_3 + p_1 - p_2 + p_3 = -1 \tag{18}$$

and

$$-\frac{1}{10}p_3^2 + 2p_2 - 2p_3 = -2. \tag{19}$$

From eq. (17) we obtain two values of p_2 namely -1.4 and 21.4. We choose $p_2 = 21.4$. Next, from eq. (19) we obtain two values of p_3, but we choose $p_3 = 13.4$.
Finally from eq. (18) we get $p_1 = 35.6$. Therefore

$$P = \begin{bmatrix} 35.6 & 21.4 \\ 21.4 & 13.4 \end{bmatrix}. \tag{20}$$

The controller is obtained from $u = -R^{-1}B^T P x$, which gives

$$u = \begin{bmatrix} -2.14 & -1.34 \end{bmatrix} x \tag{21}$$

(d) (12 points) Suppose now that the industrial process is hazardous. Therefore, it is not possible to have measurements of all the states. Design a state observer to estimate the internal states. It is required that the observer has an exponential convergence rate of at least 10, that is $|x(t) - \hat{x}(t)| \leq |x(0) - \hat{x}(0)| \exp(-10t)$.

Solution: Let

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \tag{22}$$

The matrix $A - LC$ reads as

$$A - LC = \begin{bmatrix} -l_1 & 1 \\ 1 - l_2 & -1, \end{bmatrix} \tag{23}$$

and has a characteristic polynomial

$$p(s) = s^2 + (1 + l_1)s + l_1 + l_2 - 1 \tag{24}$$

To design the observer, let $\alpha > 10$, $\beta > 10$ and define a desired polynomial as $p_d(s) = (s + \alpha)(s + \beta) = s^2 + (\alpha + \beta)s + \alpha\beta$. To obtain l_1 and l_2 from $p(s) = p_d(s)$. By the choice of α and β we guarantee that the convergence rate is at least 10. The we obtain

$$\begin{aligned} l_1 &= \alpha + \beta - 1 \\ l_2 &= 2 + \alpha\beta - \alpha - \beta \end{aligned} \tag{25}$$

*Note: students may choose arbitrary numerical values for α and β as long as they are greater that 10.

(e) (4 points) Finally, assume that you want to implement the optimal state feedback controller using the estimated states. Is this possible?

If yes, why and how would you do it?

If not, why and what would you propose as a solution?

Motivate your answer and provide enough justification.

Solution: Yes it is possible, the justification is the separation principle. Once we have designed the state observer and the feedback controller, we just substitute the feedback control law $u = -Kx$ by $u = -K\hat{x}$. If we then write the combined error dynamics, we get the system

$$\begin{bmatrix} \dot{e}_c \\ \dot{e}_o \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} e_c \\ e_o \end{bmatrix} \tag{26}$$

where e_c denotes the error for the control system and e_o denotes the estimation error. Since the matrices $A - BK$ and $A - LC$ are stable, the overall error dynamics converge to 0.

3. (35 TOTAL points) Consider the mechanical system shown in fig. 3

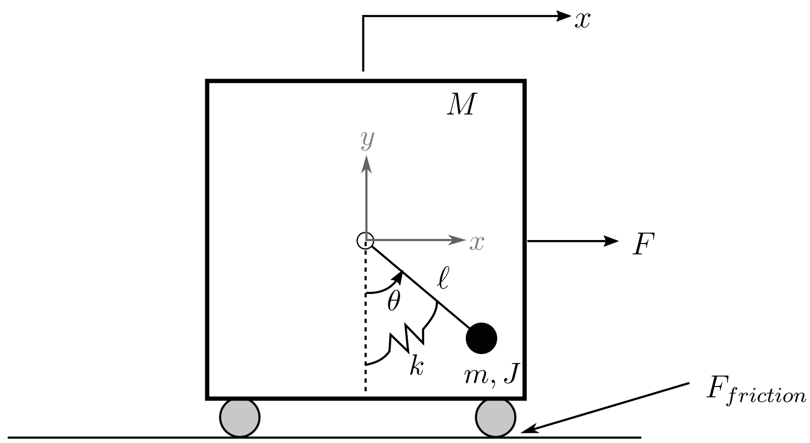


Figure 3: A mechanical system

where M is the mass of the cart, m the mass of the pendulum, J the moment of inertia of the pendulum, ℓ the length of the pendulum, x denotes the displacement of the cart from the rest position, θ is the angle of the pendulum measured from its rest position, F denotes an external force applied to the cart, and $F_{friction}$ denotes the friction force between the wheels of the cart and the ground, and is given by $F_{friction} = b\dot{x}_M$. It is assumed that the pendulum is connected to a spring with coefficient k and with force $f(\theta) = k\theta$.

(a) (1 point) Denote by (x_M, y_M) the position of the cart and by (x_m, y_m) the position of the pendulum. Suppose that the reference frame is located at the pendulum's pivot point when the system is at rest. Write down the cart's position (x_M, y_M) and the pendulum's position (x_m, y_m) in terms of the variables (x, θ).

Solution:

- The horizontal position of the cart is $x_M = x$.
- The vertical position of the cart is $y_M = 0$.
- The horizontal position of the pendulum is $x_m = x + \ell \sin\theta$.
- The vertical position of the pendulum is $y_m = -\ell \cos\theta$

(b) (10 points) Write down the Kinetic and Potential Energies of the system, and the Lagrangian function.

Hint: the contribution of the rotational spring to the potential energy is similar to a linear spring. If $f(\theta)$ is the force exerted by the spring, then its (stored) potential energy is $E_{spring} = \int_0^\theta f(s)ds$.

Solution: The total kinetic energy is given by the velocities contribution of the masses and the inertia, that is:

$$
\begin{aligned}
K &= \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 + \frac{1}{2}J\omega^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left((\dot{x} + \ell\cos\theta\dot{\theta})^2 + (\ell\sin\theta\dot{\theta})^2\right) + \frac{1}{2}J\dot{\theta}^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\ell\cos\theta\dot{x}\dot{\theta} + \ell^2\cos^2\theta\dot{\theta}^2 + \ell^2\sin^2\theta\dot{\theta}^2\right) + \frac{1}{2}J\dot{\theta}^2 \\
&= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\cos\theta\dot{x}\dot{\theta} + \frac{1}{2}J\dot{\theta}^2.
\end{aligned}
\tag{27}
$$

On the other hand, the potential energy of the system is given by

$$
U = \frac{1}{2}k\theta^2 - mg\ell\cos\theta. \tag{28}
$$

Therefore, the Lagrangian function reads as

$$
L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(m\ell^2 + J)\dot{\theta}^2 + m\ell\cos\theta\dot{x}\dot{\theta} - \frac{1}{2}k\theta^2 + mg\ell\cos\theta \tag{29}
$$

(c) (10 points) Obtain the equations of motion from the Lagrangian function obtained in part (b). Recall that the equations of motion are obtained via the formula

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = Bu - \frac{\partial D}{\partial \dot{q}}, \tag{30}
$$

where L is the Lagrangian, D the Rayleigh dissipation function, and B the input matrix.

Solution: We have:

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial \theta} &= -m\ell\sin\theta\dot{x}\dot{\theta} - mg\ell\sin\theta - k\theta \\
\frac{\partial L}{\partial \dot{x}} &= (M+m)\dot{x} + m\ell\cos\theta\dot{\theta} \\
\frac{\partial L}{\partial \dot{\theta}} &= (m\ell^2 + J)\dot{\theta} + m\ell\cos\theta\dot{x}
\end{aligned}
\tag{31}
$$

It follows that

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= (M+m)\ddot{x} + m\ell\cos\theta\ddot{\theta} - m\ell\sin\theta\dot{\theta}^2 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= (m\ell^2+J)\ddot{\theta} + m\ell\cos\theta\ddot{x} - m\ell\sin\theta\dot{\theta}\dot{x}\end{aligned} \tag{32}$$

Finally we get

$$\begin{aligned}(M+m)\ddot{x} + m\ell\cos\theta\ddot{\theta} - m\ell\sin\theta\dot{\theta}^2 &= F - b\dot{x} \\ (m\ell^2+J)\ddot{\theta} + m\ell\cos\theta\ddot{x} + mg\ell\sin\theta + k\theta &= 0\end{aligned} \tag{33}$$

(d) (9 points) Linearize the equations of motion obtained in part (c) around the equilibrium point $(x^*, \theta^*, \dot{x}^*, \dot{\theta}^*) = (0, 0, 0, 0)$, and write down a state-space representation of the linearized system assuming that the measured output is the position of the cart.
Hint: First, write the equations of motion of part (c) in the following format:

$$M(x,\theta)\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} + C(x,\theta,\dot{x},\dot{\theta})\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} + G(x,\theta) = \begin{bmatrix}F\\ 0\end{bmatrix}, \tag{34}$$

where $M(x,\theta)$ is a 2×2 symmetric, positive-definite, invertible matrix, $C(x,\theta,\dot{x},\dot{\theta})$ is a 2×2 matrix, and $G(x,\theta)$ is a 2×1 vector. Next, linearize the nonlinear terms of $M(x,\theta)$, $C(x,\theta,\dot{x},\dot{\theta})$, and $G(x,\theta)$ around the equilibrium point. Finally choose appropriate state variables to come up with the state-space representation.

Solution: Recall that $\sin\theta \sim \theta$ and $\cos\theta \sim 1$ near the equilibrium point. Therefore, the linearized system reads as

$$\begin{aligned}(M+m)\ddot{x} + m\ell\ddot{\theta} &= F - b\dot{x} \\ (m\ell^2+J)\ddot{\theta} + m\ell\ddot{x} + mg\ell\theta + k\theta &= 0\end{aligned} \tag{35}$$

For convenience, let us write the previous equation in vector form as

$$\begin{bmatrix}M+m & m\ell\\ m\ell & m\ell^2+J\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = -\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & k+mg\ell\end{bmatrix}\begin{bmatrix}x\\ \theta\end{bmatrix} + \begin{bmatrix}F\\ 0\end{bmatrix} \tag{36}$$

or equivalently

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}M+m & m\ell\\ m\ell & m\ell^2+J\end{bmatrix}^{-1}\left(-\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & k+mg\ell\end{bmatrix}\begin{bmatrix}x\\ \theta\end{bmatrix} + \begin{bmatrix}F\\ 0\end{bmatrix}\right) \tag{37}$$

Again, for convenience let

$$\begin{aligned}\begin{bmatrix}M+m & m\ell\\ m\ell & m\ell^2+J\end{bmatrix}^{-1} &= \frac{1}{(M+m)(m\ell^2+J) - m^2\ell^2}\begin{bmatrix}m\ell^2+J & -m\ell\\ -m\ell & M+m\end{bmatrix} \\ &= \begin{bmatrix}\alpha_1 & \alpha_2\\ \alpha_2 & \alpha_3\end{bmatrix}\end{aligned} \tag{38}$$

Then we have that eq. (37) reads as

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = -\begin{bmatrix}\alpha_1 b & 0\\ \alpha_2 b & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} - \begin{bmatrix}0 & \alpha_2(k+mg\ell)\\ 0 & \alpha_3(k+mg\ell)\end{bmatrix}\begin{bmatrix}x\\ \theta\end{bmatrix} + \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}F. \tag{39}$$

Next, define state variables $X = (X_1, X_2, X_3, X_4) = (x, \theta, \dot{x}, \dot{\theta})$. Note that x does not appear in the equations of motion, so it is not necessarily a state variable. But since we are told to choose the position of the cart as the output, we must include x in the state variables. It follows that

$$\begin{aligned}\dot{X} = \begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\\ \dot{X}_4\end{bmatrix} &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -\alpha_2(k+mg\ell) & -\alpha_1 b & 0\\ 0 & -\alpha_3(k+mg\ell) & -\alpha_2 b & 0\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \alpha_1\\ \alpha_2\end{bmatrix}F\\ y &= \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}X.\end{aligned} \tag{40}$$

(e) (5 points) Obtain a new but *equivalent* state space representation of the linear system obtained in part (d). Is the input-output behavior of the new system equal to the input-output behavior of the linear system obtained in (d)? Provide your arguments.

Solution: Let eq. (40) be denoted as

$$\begin{aligned}\dot{X} &= AX + Bu\\ y &= CX,\end{aligned} \tag{41}$$

and let T be a non-singular 2×2 matrix. Let $Z = TX$. In terms of the new variable Z, eq. (41) reads as

$$\begin{aligned}\dot{Z} &= TAT^{-1}Z + TBu\\ y &= CT^{-1}Z.\end{aligned} \tag{42}$$

The input-output behavior of eq. (41) and of eq. (42) are the same since they have the same transfer function. To see this fact, the trasnfer function of eq. (42) is given by

$$\begin{aligned}G(s) &= CT^{-1}\left(sI - TAT^{-1}\right)^{-1}TB\\ &= C\left(T^{-1}(sI - TAT^{-1})T\right)^{-1}B\\ &= C(sI - A)^{-1}B,\end{aligned} \tag{43}$$

which is the transfer function of eq. (41).

4. (10 TOTAL points) Consider the problem of dynamical modeling of an electro-hydraulic system for an assisted steering wheel in a car. It is a multi-domain system which consists of mechanical system, fluid system and electrical system as shown in fig. 4. The Voltage source V_s is used to rotate the valve through an electro-mechanical coupling device which has the relation of

$$\omega = \alpha V_{coupling},\ i_L = \alpha T, \tag{44}$$

where α is the coupling constant, ω is the angular velocity of the valve, i_L is the current through the inductor and T is the torque applied to the valve. The moment of inertia of the valve is denoted by J. The angular position of the valve θ determines the flow rate Q_m based of the following relation:

$$P_{12} = f(\theta)Q_m^2, \tag{45}$$

where f is a nonlinear function that depends on the valve position θ . The pressure source P_s provides a constant pressure. Based on the pressure across the hydraulic motor P_{2r} , the hydraulic motor generates a force F that drives a mechanical system with mass m and which is connected to a spring and a damper with constants k and b , respectively. The displacement of the mechanical system is denoted by x and the velocity is denoted by v. The fluid mechanical coupling device (or the hydraulic motor) satisfies

$$F = DP_{2r},\ Dv = Q_m, \tag{46}$$

where D is the coupling constant of the hydraulic motor. Derive the state equations of the full system with the pressure P_s as the input and the velocity v as the measured output.

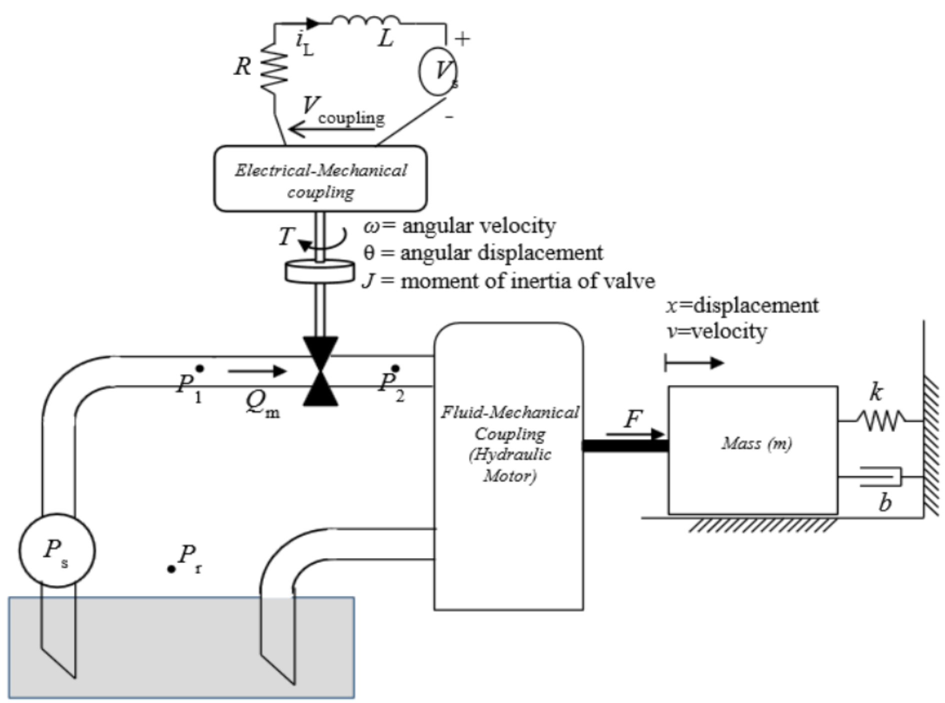


Figure 4: A simplified electro-hydro system of an assisted steering wheel in a car.

Solution: From the mechanical side we have $m\ddot{x} = F - kx - b\dot{x} = DP_{2r} - kx - b\dot{x}$ and $J\dot{\omega} = T = \frac{i_L}{\alpha}$. From the electrical side we have $V_R + V_L + V_s - V_{coupling=0}$, which implies $L\frac{di_L}{dt} = -Ri_L - V_s + \frac{\omega}{\alpha}$. From the hydraulic side we have $P_s = P_{12} + P_{2r}$, which implies $P_{2r} = P_s - f(\theta)Q_m^2 = P_s - f(\theta)D^2v^2$. Next we choose as state variables $(\theta, \omega, x, v, i_L)$. Thus we have

$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= \frac{1}{J}\frac{i_L}{\alpha} \\
\dot{x} &= v \\
\dot{v} &= \frac{D}{m}(P_s - f(\theta)D^2v^2) - \frac{k}{m}x - \frac{b}{m}v \\
\dot{i}_L &= -\frac{R}{L}i_L - \frac{1}{L}V_s + \frac{1}{\alpha L}\omega
\end{aligned}
\tag{47}
$$

Then the state space representation reads as

$$
\begin{aligned}
\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{x} \\ \dot{v} \\ \dot{i}_L \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & {}^1/_{J\alpha} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^k/_m & -{}^b/_m & 0 \\ 0 & {}^1/_{\alpha L} & 0 & 0 & -{}^R/_L \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ x \\ v \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -{}^D/_m f(\theta)D^2v^2 \\ -{}^{V_s}/_L \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ {}^D/_m \\ 0 \end{bmatrix} P_s \\
y &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ x \\ v \\ i_L \end{bmatrix}
\end{aligned}
\tag{48}
$$

5. (15 TOTAL + 5 bonus points) Consider a plant described by the transfer function

$$G(s) = \frac{0.5}{s^2 + 3s + 2}, \tag{49}$$

and a proportional controller with transfer function $C(s) = 2$. The loop gain is defined as $L(s) = G(s)C(s)$. The Nyquist plot of $L(s)$ is shown in fig. 5.

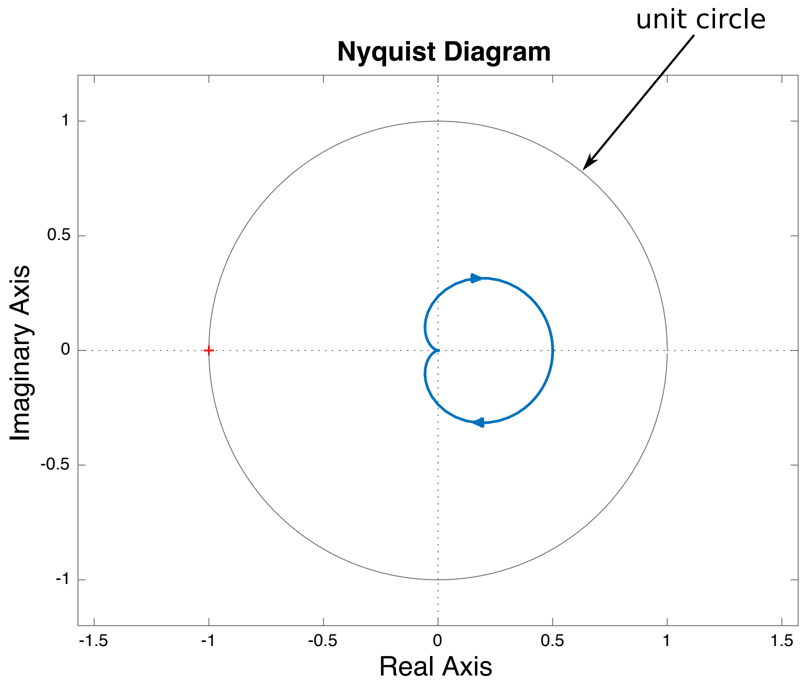


Figure 5: Nyquist plot of the loop gain $L(s) = G(s)C(s)$

(a) (5 points) What is the phase margin of the loop gain?

Hint: You can obtain the answer either from fig. 5 or by remembering that the phase margin is computed as $PM = -\pi - \angle L(j\omega^*)$, where ω^* is the frequency at which the norm of $L(s)$ is equal to 1.

> **Solution:** From fig. 5 we see that the plot of $L(j\omega)$ does not intersect the unit circle, in other words $|L(j\omega)| = |G(j\omega)C(j\omega| = 1$ is never satisfied. This means that the phase margin is infinite.

(b) (3 bonus points) Give a physical interpretation of the result obtained in (a). In particular, what can you say about the stability of the system under time-delays?

> **Solution:** The system is robust under time-delays, that is, the stability of the system is not affected by time delays.

(c) (5 points) Suppose that now, a control engineer suggests a new controller with transfer function $\bar{C}(s) = \sqrt{72}$. The Nyquist plot of the loop gain $\bar{L}(s) = G(s)\bar{C}(s)$ is shown in fig. 6. What is the phase margin of $\bar{L}(s)$?

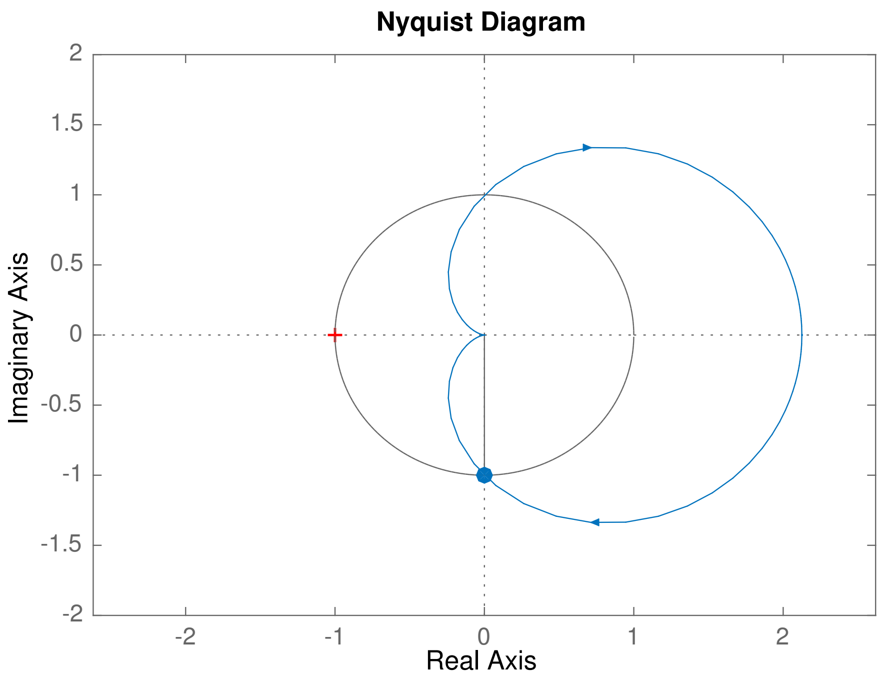


Figure 6: Nyquist plot of the loop gain $\bar{L}(s) = G(s)\bar{C}(s)$

Solution: From the plot we see that the phase margin is $-\frac{\pi}{2}$. However, this can be also computed as follows.
First obtain ω^* such that $|G(j\omega^*)\bar{C}(j\omega^*)| = 1$. We have

$$\begin{aligned}\frac{|0.5\sqrt{72}|}{|(j\omega^*)^2 + 3(j\omega^*) + 2|} &= 1 \\ \frac{0.25(72)}{(-(\omega^*)^2 + 2)^2 + 9(\omega^*)^2} &= 1 \\ \frac{18}{(\omega^*)^4 + 5(\omega^*)^2 + 4} &= 1\end{aligned} \tag{50}$$

which leads to the equation

$$(\omega^*)^4 + 5(\omega^*)^2 - 14 = 0. \tag{51}$$

Solving for a positive and real ω^* we find $\omega^* = \sqrt{2}$. Next we find the phase of the loop gain when $\omega = \omega^*$, that is

$$\begin{aligned}\angle G(j\omega^*)C(j\omega^*) &= \angle\left(\frac{0.5\sqrt{72}}{(j\sqrt{2})^2 + 3(j\sqrt{2}) + 2}\right) \\ &= \angle\left(\frac{\sqrt{72}}{6\sqrt{2}j}\right) = -\frac{\pi}{2}.\end{aligned} \tag{52}$$

Finally using the formula $PM = -\pi - \angle L(j\omega^*)$ we get

$$PM = -\pi - \left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} \tag{53}$$

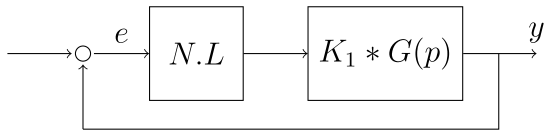
(d) (5 points) What is the maximum time delay that it is allowed in the new system such that the dynamics remain stable?

Solution:
We have obtained that the phase margin is $-\frac{\pi}{2}$, and we know that a time delay shifts the angle $\angle L(s)$ along the unit circle by $T_d\omega^*$. This means that the time delay should shift $\angle L(s)$ by less that $-\frac{\pi}{2}$, that is $T_d\omega^* < \frac{\pi}{2}$. Therefore $T_d < \frac{\pi}{2\sqrt{2}} \sim 1.1$s

(e) (2 bonus points) Compare the system in part (a) with the system in part (c), and describe the difference you notice. What can you say about their behavior under time-delays? Is one or the other better in that regard?

Solution: Introducing a higher proportional gain of the controller compromises the robustness with respect to time delays, meaning that, with respect to the ability to handle time delays, system in part (a) is better that system in part (c).

6. (5 bonus points) In the non-linear system below



the plant is stable and the Nyquist plot of the loop gain $L(s) = K_1 G(s)$ is shown in fig. 7

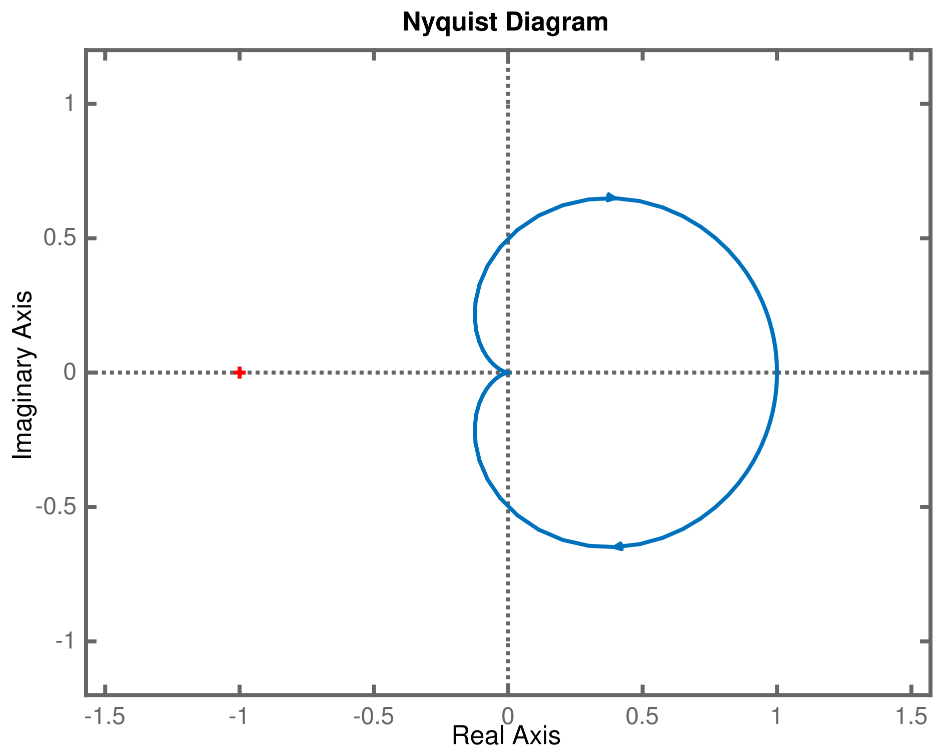


Figure 7: Nyquist plot of $K_1 G(s)$

The nonlinearity NL is given by a nonlinear sector with boundaries $[k_1, k_2]$. It is known that, due to the behavior of the nonlinearity, the boundaries satisfy $k_1 < 0$ and $k_2 > 0$. Choose appropriate bounds k_1 and k_2 such that the origin of the closed loop system is globally asymptotically stable. Motivate your answer.

Solution: From the circle criteria, we must choose the boundaries k_1, k_2 in such a way that the disk $D(k_1, k_2)$ contains the Nyquist plot. This is done by choosing for example $-1 < k_1 < 0$ and $0 < k_2 < 2$.

*Students may choose any numerical value that satisfies the above argument.